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Kepler's Problem in Rotating Reference Frames Part 2: Relative Orbital Motion

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A direct application of Kepler's problem in rotating reference frames is the orbital relative motion study. The nonlinear differential equation modeling the motion is solved by means of tensorial and vectorial regularization methods. The general framework for obtaining exact solutions to the relative orbital motion is given when the reference trajectory is elliptic, parabolic, and hyperbolic. All the results are presented in a vectorial closed form, frame independent and without singularities. The case of circular reference trajectory is particularized with complete vectorial and parametric solutions. It is proved that the solution to Hill–Clohessy–Wiltshire equations is the first linear approximation of the exact solution presented in this paper.

I. Introduction

THIS paper is concerned with the relative orbital motion, an important application of the solution to Kepler's problem in rotating reference frames. It is a subject of significant practical interest. In the last few years, space programs involving satellite formation flying were sponsored by several governmental agencies. Applications to spacecraft formation flying are multiple. Long baseline interferometry, stereographic images, or synthetic aperture radar are relevant examples. Proper positioning of the satellites in the clusters is essential in most missions.

The most common approach to spacecraft relative motion uses the Hill–Clohessy–Wiltshire (HCW) linear model introduced in 1960 [1]. This model assumes small deviations from a circular reference orbit. Because the HCW model is limited to short-term motion, the approach was extended by several authors [2–4]. Other researchers have modeled the relative motion starting from the HCW equations and taking into account various orbital perturbations: oblateness [5–8] and drag [9]. An important step forward is the generalization of the HCW equations for an elliptic Keplerian reference orbit, that was done by Lawden [10] and Tschauner and Hempel [11,12]. The results for the elliptic reference trajectory were studied and extended by many authors [13–24]. The solutions for the relative orbital Keplerian motion use the Cartesian initial conditions or the orbital elements as constant of motion [25–34]. Some of the approximate solutions use time [16,20], the true anomaly [17,18], or the eccentric anomaly [11,32] of the Keplerian reference orbit as independent variables.

The problem of relative orbital dynamics is an extensive field of study and it refers to satellite formation flying [35–57] or spatial rendezvous studies [58–61]. Comparative papers evaluate and analyze the accuracy of the various theories of the relative motion of satellites [62,63].

The present paper uses the solution to Kepler's problem in rotating reference frames introduced in [64] to offer an exact vectorial solution to the relative orbital motion. The results stand for any reference Keplerian trajectory (elliptic, parabolic, hyperbolic). In [64], by regularization methods, the strong nonlinear differential equation that models Keplerian motion in a rotating reference frame is transformed into a linear one with constant coefficients. The singularity for zero position vector vanishes and the solutions to this new equation are in a closed vectorial form.

This paper is structured as it follows. Section II introduces the general framework for the relative orbital motion problem, presenting the solution to the general case. Section III gives an exact solution to the orbital relative motion when the reference orbit is circular, generalizing the model and the solution offered by the HCW equations. An analogous to the classic Keplerian eccentric anomaly is used in the expressions for relative position and relative velocity vectors. Contrary to the HCW equations, that are linearized derivations from the nonlinear problem that describes the motion, we start directly from the nonlinear original differential equation that models the relative motion. The advantage is that we obtain the complete prediction of the behavior of the studied satellite in its relative motion. The case of zero angular momentum is also studied with complete solutions in all three main cases. The HCW equations offer a solution involving secular terms, so their applicability is restricted to short time intervals. The solution presented here stands for unbounded time intervals and does not have singularities. The expressions are completely vectorial and frame-independent.

Section III.D shows that the solution to HCW equations is the first linear approximation to the exact solution previously given. We offer here an interesting qualitative analysis of the motion described by the HCW equations.

The same Nomenclature as in [64] is used.

II. General Relative Keplerian Motion

A body, denoted as Chief, orbiting around a Keplerian attraction center O is considered. In an inertial reference frame with the origin in the attraction center, its trajectory will be an ellipse, a parabola, or a hyperbola. At the initial moment of time t_0, another body, denoted as Deputy, is launched from the vicinity of the Chief. The launch is made such as at time t_0 the relative position of the Deputy to the Chief is described by vector $\Delta \boldsymbol{r}$ and the relative velocity by vector $\Delta \boldsymbol{v}$. Using the results presented in [64], we will prove that an exact solution to the relative motion of the Deputy to the Chief problem may be given. This solution does not involve linearizations of the equation describing the motion, which are used in most orbital mechanics papers [1–16,65].

The trajectory of the Chief with respect to an inertial frame with the origin in the attraction center is a conic with one focus in the attraction center (see Fig. 1). Its motion along this trajectory is considered to be known. The position vector of the Chief with respect to the inertial reference frame is denoted $\boldsymbol{r}_C$, $\boldsymbol{h}_C$ is its specific angular momentum, and θ is its true anomaly.

We define three unit vectors: $\boldsymbol{i}_0 = \boldsymbol{r}_C / r_C$ in the orbit radius direction, $\boldsymbol{k}_0 = \boldsymbol{h}_C / h_C$ in the angular momentum direction, and

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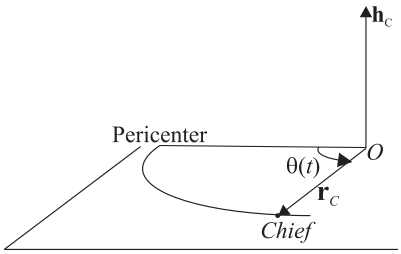

Fig. 1 The Keplerian inertial motion of the Chief.

$\boldsymbol{j}_0 = \boldsymbol{k}_0 \times \boldsymbol{i}_0$ completes a positive oriented reference frame. Related to a rotating reference frame with the origin in the attraction center $\{O, \boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$ that rotates with angular velocity $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}_0 = (1/r_C^2)\boldsymbol{h}_C$, the Chief trajectory is described by

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{0}$$
$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0$$
$$\dot{\boldsymbol{r}}(t_0) = \frac{\boldsymbol{v}_0 \cdot \boldsymbol{r}_0}{r_0^2}\boldsymbol{r}_0 \tag{1}$$

and it is rectilinear (see Fig. 2).

Vectors $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ represent the position and, respectively, the velocity of the Chief at $t = t_0$ in an inertial frame with the origin in O.

In the same reference frame, the motion of the Deputy is described by (see Fig. 3)

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{0}$$
$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0 + \Delta\boldsymbol{r}$$
$$\dot{\boldsymbol{r}}(t_0) = \frac{\boldsymbol{v}_0 \cdot \boldsymbol{r}_0}{r_0^2}\boldsymbol{r}_0 + \Delta\boldsymbol{v} \tag{2}$$

Equations (1) and (2) are related to a rotating reference frame with fixed direction angular velocity $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}_0 = (1/r_C^2)\boldsymbol{h}_C$. The solution to Eq. (1) is

$$\boldsymbol{r}_C = \frac{p_C}{1 + e_C\cos\theta}\frac{\boldsymbol{r}_0}{r_0} \tag{3}$$

where p_C is the semilatus rectum of the conic (the Chief trajectory related to the inertial reference frame) and e_C its eccentricity. As vector $\boldsymbol{\omega}$ has fixed direction, the tensorial map $\boldsymbol{R}_{-\boldsymbol{\omega}}$ has the explicit expression [64]

$$\boldsymbol{R}_{-\boldsymbol{\omega}(t)} = \exp\left(-\int_{t_0}^{t}\tilde{\boldsymbol{\omega}}(\xi)\,\mathrm{d}\xi\right) = \boldsymbol{I}_3 - \frac{\sin[\theta(t) - \theta(t_0)]}{\omega}\tilde{\boldsymbol{\omega}} + \frac{1 - \cos[\theta(t) - \theta(t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2 \tag{4}$$

As follows from the main theorem in [64], Sec. III, the solution to Eq. (2) is obtained by applying the tensorial operator $\boldsymbol{R}_{-\boldsymbol{\omega}}$ to the solution to

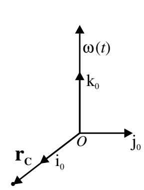

Fig. 2 Motion of the Chief in a noninertial associated reference frame is rectilinear.

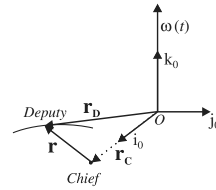

Fig. 3 The relative motion of the Deputy satellite.

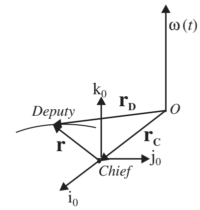

Fig. 4 The Deputy motion in the LVLH reference frame.

$$\ddot{\boldsymbol{r}} + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{0}$$
$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0 + \Delta\boldsymbol{r}$$
$$\dot{\boldsymbol{r}}(t_0) = \boldsymbol{v}_0 + \Delta\boldsymbol{v} + \boldsymbol{\omega}_0 \times \Delta\boldsymbol{r} \tag{5}$$

where $\boldsymbol{\omega}_0 = (1/r_0^2)\boldsymbol{h}_C$.

We define now the local vertical local horizontal (LVLH) or Hill reference frame, with the origin in the Chief (considered here a geometrical point) and the same unit vectors $\boldsymbol{i}_0$, $\boldsymbol{j}_0$, $\boldsymbol{k}_0$. It is the noninertial reference frame to which the orbital relative motion of the Deputy is referred (see Fig. 4). The relative motion is described by

$$\boldsymbol{r} = \boldsymbol{r}_D - \boldsymbol{r}_C \Rightarrow \tag{6}$$

$$\boldsymbol{r} = \boldsymbol{R}_{-\boldsymbol{\omega}}\boldsymbol{r}_* - \frac{p_C}{1 + e_C\cos\theta}\frac{\boldsymbol{r}_0}{r_0} \tag{7}$$

where $\boldsymbol{r}_C$ is the solution to Eq. (1), $\boldsymbol{r}_D$ the solution to Eq. (2), and $\boldsymbol{r}_*$ the solution to Eq. (5).

From Eq. (7), it follows that the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ is computed by

$$\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{R}_{-\boldsymbol{\omega}}\boldsymbol{r}_*) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_C}{1 + e_C\cos\theta}\right)\frac{\boldsymbol{r}_0}{r_0} \tag{8}$$

By making the computations in Eq. (8) and by using the properties of the tensorial map $\boldsymbol{R}_{-\boldsymbol{\omega}}$, it follows

$$\boldsymbol{v} = \boldsymbol{R}_{-\boldsymbol{\omega}}\dot{\boldsymbol{r}}_* - \tilde{\boldsymbol{\omega}}\boldsymbol{R}_{-\boldsymbol{\omega}}\boldsymbol{r}_* - \frac{\dot{\theta}e_C p_C\sin\theta}{(1 + e_C\cos\theta)^2}\frac{\boldsymbol{r}_0}{r_0} \tag{9}$$

From Eqs. (7) and (9), by taking Eq. (4) into account, it follows that the motion of the Deputy with respect to the LVLH frame is described by

$$\boldsymbol{r}(t) = \frac{\boldsymbol{\omega} \cdot \boldsymbol{r}_*}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\theta(t) - \theta(t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{r}_* - \frac{\cos[\theta(t) - \theta(t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{r}_* - \frac{p_C}{1 + e_C\cos\theta}\frac{\boldsymbol{r}_0}{r_0} \tag{10}$$

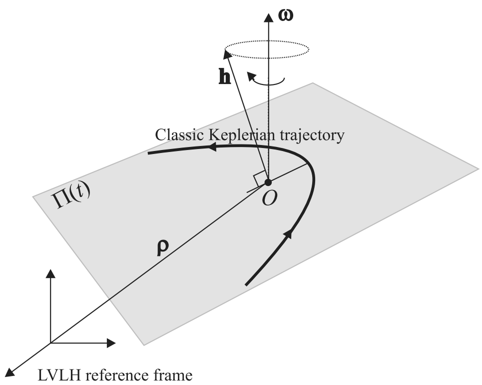

Fig. 5 Relative orbital motion: geometric interpretation.

$$\begin{aligned}\boldsymbol{v}(t) &= \frac{\boldsymbol{\omega}\cdot\dot{\boldsymbol{r}}_*}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\theta(t)-\theta(t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\dot{\boldsymbol{r}}_* \\ &- \frac{\cos[\theta(t)-\theta(t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\dot{\boldsymbol{r}}_* - \frac{h_C(1+e_C\cos\theta)^2}{p_C^2} \\ &\times\left[\frac{\cos[\theta(t)-\theta(t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{r}_* - \frac{\sin[\theta(t)-\theta(t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{r}_*\right] \\ &- \frac{h_C e_C\sin\theta}{p_C}\frac{\boldsymbol{r}_0}{r_0}\end{aligned} \tag{11}$$

where $\boldsymbol{r}_*$ is the solution to the inertial problem (5).

Using the algorithm presented in [64], Subsection IV.B.2, explicit solutions and then approximate solutions with time, true anomaly or eccentric anomaly as independent variables may be deduced.

Remark 1: From Eq. (7) a suggestive geometric interpretation for the Keplerian relative motion with respect to the LVLH reference frame is revealed. The relative Keplerian motion is composed from three different types of motion (see also Fig. 5):

1) A Keplerian classic motion described by Eq. (5), situated in a plane $\boldsymbol{\Pi}(t)$ or on a straight line.

2) A precession with angular velocity $-\boldsymbol{\omega}$ around a point O of the plane (straight line) where the classic Keplerian motion takes place. Point O represents one of the foci of this conical trajectory.

3) A rectilinear motion of point O described by:

$$\rho = -\frac{p_C}{1+e_C\cos\theta}\frac{\boldsymbol{r}_0}{r_0} \tag{12}$$

When the reference trajectory is elliptic, the rectilinear motion of point O is periodic. Its period equals the period of the Keplerian inertial reference motion. If the Keplerian motion in plane $\boldsymbol{\Pi}(t)$ is also periodic and the ratio of the two main periods is rational, then the trajectory is a closed curve (orbit resonance). This represents the necessary and sufficient condition for the relative motion to be periodic. In other words, the periodicity of the relative motion is equivalent with the commensurability between the period of the reference motion and the period of motion in plane $\boldsymbol{\Pi}(t)$.

When the reference trajectory is circular, point O is stationary with respect to LVLH frame.

A forthcoming paper will present the time-explicit solutions for relative Keplerian dynamics in case the reference trajectory is elliptic, parabolic, hyperbolic. These solutions generalize the approximate results obtained from the HCW equations [1], Lawden [10], and Tschauner–Hempel equations [11,12]. In fact, the approximate solutions derived from the linearized models used by various authors are the first linear approximation of the exact solution offered by Eqs. (10) and (11).

III. Orbital Relative Motion: Circular Reference Trajectory

When the reference trajectory is circular, vector $\boldsymbol{\omega}$ is constant, $\boldsymbol{\omega} = \boldsymbol{\omega}(t_0)$. Then Eq. (1) becomes

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega}\times\dot{\boldsymbol{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{0}, \quad \boldsymbol{r}(t_0) = \boldsymbol{r}_0, \quad \dot{\boldsymbol{r}}(t_0) = \boldsymbol{0} \tag{13}$$

and it has a constant solution $\boldsymbol{r}(t) = \boldsymbol{r}_0$, $t \geq t_0$. It is realized when additional conditions

$$r_0 = \left(\frac{\mu}{\omega^2}\right)^{1/3}; \qquad \boldsymbol{r}_0\cdot\boldsymbol{\omega} = 0; \qquad \boldsymbol{\omega} = \overrightarrow{\text{constant}} \tag{14}$$

are satisfied. In the rotating reference frame $\{O, \boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$, the motion of the Deputy is described by

$$\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega}\times\dot{\boldsymbol{\rho}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}) + \frac{\mu}{\rho^3}\boldsymbol{\rho} = \boldsymbol{0}, \qquad \boldsymbol{\rho}(t_0) = \boldsymbol{r}_0 + \Delta\boldsymbol{r}, \qquad \dot{\boldsymbol{\rho}}(t_0) = \Delta\boldsymbol{v} \tag{15}$$

The Cauchy problem (15) represents the perturbation of Eq. (13) by changing the initial conditions

$$\boldsymbol{r}_0 \to \boldsymbol{r}_0 + \Delta\boldsymbol{r}; \qquad \boldsymbol{0} \to \Delta\boldsymbol{v} \tag{16}$$

As in the preceding section, the relative law of motion of the Deputy with respect to the Chief is given by

$$\boldsymbol{r} = \boldsymbol{R}_{-\boldsymbol{\omega}(t)}\boldsymbol{r}_* - \boldsymbol{r}_0 \tag{17}$$

where $\boldsymbol{r}_*$ is the solution to

$$\ddot{\boldsymbol{r}} + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{0}, \qquad \boldsymbol{r}(t_0) = \boldsymbol{r}_0 + \Delta\boldsymbol{r}, \qquad \dot{\boldsymbol{r}}(t_0) = \boldsymbol{\omega}\times(\boldsymbol{r}_0 + \Delta\boldsymbol{r}) + \Delta\boldsymbol{v} \tag{18}$$

The prime integrals of (18) are

$$\boldsymbol{h}_0 = (\Delta\boldsymbol{r} + \boldsymbol{r}_0)\times[\Delta\boldsymbol{v} + \boldsymbol{\omega}\times(\Delta\boldsymbol{r} + \boldsymbol{r}_0)] \tag{19}$$

$$\xi = \frac{1}{2}[\Delta\boldsymbol{v} + \boldsymbol{\omega}\times(\Delta\boldsymbol{r} + \boldsymbol{r}_0)]^2 - \frac{\mu}{|\Delta\boldsymbol{r} + \boldsymbol{r}_0|} \tag{20}$$

$$\boldsymbol{e}_0 = \frac{[\Delta\boldsymbol{v} + \boldsymbol{\omega}\times(\Delta\boldsymbol{r} + \boldsymbol{r}_0)]\times\boldsymbol{h}_0}{\mu} - \frac{\Delta\boldsymbol{r} + \boldsymbol{r}_0}{|\Delta\boldsymbol{r} + \boldsymbol{r}_0|} \tag{21}$$

As vector $\boldsymbol{\omega}$ is constant, we use (4) for computing $\boldsymbol{R}_{-\boldsymbol{\omega}(t)}$. Making all the computations in (17), it follows that

$$\boldsymbol{r}(t) = \frac{\boldsymbol{r}_*\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\boldsymbol{\omega}\times\boldsymbol{r}_* - \frac{\cos[\omega(t-t_0)]}{\omega^2}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_*) - \boldsymbol{r}_0 \tag{22}$$

where $\boldsymbol{r}_*$ is the solution to Eq. (18).

If the specific energy $\xi < 0$, then the trajectory described by (22) is bounded; if $\xi \geq 0$, the trajectory may be unbounded, depending on the initial conditions.

According to the study made in [64], Sec. IV, and *Remark 1* from Sec. II of the present paper, the motion of the Deputy with respect to the LVLH frame can be decomposed into a classic Keplerian motion and a regular precession with angular velocity $-\boldsymbol{\omega}$. The trajectory is bounded or unbounded, depending on the initial conditions. When $\xi < 0$, $\boldsymbol{h}_0 \neq 0$, the trajectory is bounded and it may be decomposed into a classic elliptic Keplerian motion and a regular precession with angular velocity $-\boldsymbol{\omega}$. If $\xi < 0$, $\boldsymbol{h}_0 = 0$, the trajectory may be decomposed into a rectilinear Keplerian motion and a regular precession with angular velocity $-\boldsymbol{\omega}$. When $\xi \geq 0$, $\boldsymbol{h}_0 \neq 0$, the trajectory is unbounded and it may be decomposed in a classic parabolic or hyperbolic Keplerian motion in a plane $\boldsymbol{\Pi}(t)$ and the regular precession of plane $\boldsymbol{\Pi}(t)$ with angular velocity $-\boldsymbol{\omega}$. When $\xi \geq 0$, $\boldsymbol{h}_0 = \boldsymbol{0}$, the motion may be decomposed into a rectilinear Keplerian motion and a regular precession with angular velocity $-\boldsymbol{\omega}$

of the straight line associated with the classic Keplerian motion. If the trajectory is bounded, the motion lasts a finite time period, until the Deputy reaches the attraction center.

All results presented next are deduced by using the algorithm in [64], Subsection IV.B.2.

A. Negative Specific Energy: $\xi = \frac{1}{2}[\Delta \boldsymbol{v} + \boldsymbol{\omega} \times (\Delta \boldsymbol{r} + \boldsymbol{r}_0)]^2 - \frac{\mu}{|\Delta \boldsymbol{r} + \boldsymbol{r}_0|} < 0$

1. Nonzero Angular Momentum, Nonzero Eccentricity $\boldsymbol{h}_0 \neq 0$, $e \neq 0$

By applying the results from [64], Subsection IV.B.2, the trajectory is bounded.

Using

$$\boldsymbol{a}_0 = \frac{\mu}{2e|\xi|}\boldsymbol{e}_0; \qquad \boldsymbol{b}_0 = \frac{1}{e\sqrt{2|\xi|}}\boldsymbol{h}_0 \times \boldsymbol{e}_0; \qquad n = \frac{(2|\xi|)^{3/2}}{\mu}$$

and the function $E(t)$ given by

$$E(t) - e\sin E(t) = n(t - t_0) + E_0 - e\sin E_0, \qquad t \in [t_0, +\infty),$$
$$\cos E_0 = \frac{1}{e}\left(1 - n\frac{|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}{\sqrt{2|\xi|}}\right),$$
$$\sin E_0 = n\frac{\Delta\boldsymbol{v}\cdot(\boldsymbol{r}_0 + \Delta\boldsymbol{r})}{2e|\xi|}\cdot\left[1 - \frac{\boldsymbol{\omega}\cdot\boldsymbol{h}_0}{\mu}|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|\right]$$

it follows that for $t \in [t_0, +\infty)$

$$\boldsymbol{r}(t) = [\cos E(t) - e]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + \sin E(t)\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0\right\} - \boldsymbol{r}_0 \tag{23}$$

$$\boldsymbol{v}(t) = \frac{-n\sin E(t)}{1 - e\cos E(t)}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + \frac{n\cos E(t)}{1 - e\cos E(t)} \times \left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0\right\} + [\cos E(t) - e]\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0\right\} + \sin E(t)\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0\right\} \tag{24}$$

Written in the LVLH reference frame $\{\boldsymbol{i}_0 = \boldsymbol{r}_0/r_0, \boldsymbol{j}_0 = (\omega \times \boldsymbol{r}_0)/\omega r_0, \boldsymbol{k}_0 = \omega/\omega\}$

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i}_0 + y(t)\boldsymbol{j}_0 + z(t)\boldsymbol{k}_0$$

The parametric equations of the trajectory become

$$x(t) = [\cos E(t) - e]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \sin E(t)\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} - r_0 \tag{25}$$

$$y(t) = [\cos E(t) - e]\left\{-\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} + \sin E(t)\left\{-\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} \tag{26}$$

$$z(t) = [\cos E(t) - e]\frac{\boldsymbol{a}_0\cdot\omega}{\omega} + \sin E(t)\frac{\boldsymbol{b}_0\cdot\omega}{\omega} \tag{27}$$

The parametric equations of the velocity become

$$\dot{x}(t) = \frac{n\sin E(t)}{1 - e\cos E(t)}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \frac{n\cos E(t)}{1 - e\cos E(t)}\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + [\cos E(t) - e] \times \left\{-\omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} + \sin E(t)\left\{-\omega\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} \tag{28}$$

$$\dot{y}(t) = \frac{n\sin E(t)}{1 - e\cos E(t)}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] - \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} + \frac{n\cos E(t)}{1 - e\cos E(t)} \times \left\{-\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} - [\cos E(t) - e]\left\{\frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right\} - \sin E(t)\left\{\frac{(\boldsymbol{b}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right\} \tag{29}$$

$$\dot{z}(t) = \frac{-n\sin E(t)}{1 - e\cos E(t)}\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega} + \frac{n\cos E(t)}{1 - e\cos E(t)}\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega} \tag{30}$$

where $t \in [t_0, +\infty)$.

2. Nonzero Angular Momentum, Zero Eccentricity $\boldsymbol{h}_0 \neq 0$, $e = 0$

All computations made in the preceding section change by making $e = 0$. Additional relations in this case are

$$\boldsymbol{a}_0 = \boldsymbol{r}_0 + \Delta\boldsymbol{r}; \qquad \boldsymbol{b}_0 = \frac{\boldsymbol{\omega}\times(\boldsymbol{r}_0 + \Delta\boldsymbol{r}) + \Delta\boldsymbol{v}}{n};$$
$$E(t) = n(t - t_0); \qquad n = \frac{(2|\xi|)^{\frac{3}{2}}}{\mu} \tag{31}$$

with $a_0 = b_0$.

$$\boldsymbol{r}(t) = \cos[n(t-t_0)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + \sin[n(t-t_0)]\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0\right\} - \boldsymbol{r}_0 \tag{32}$$

$$\boldsymbol{v}(t) = -n\sin[n(t-t_0)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + n\cos[n(t-t_0)] \times \left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0\right\} + \cos[n(t-t_0)]\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0\right\} + \sin[n(t-t_0)]\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0\right\} \tag{33}$$

The parametric equations of the trajectory become

$$x(t) = \cos[n(t-t_0)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \sin[n(t-t_0)] \times \left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} - r_0 \tag{34}$$

$$y(t) = \cos[n(t-t_0)]\left\{-\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} + \sin[n(t-t_0)] \times \left\{-\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} \tag{35}$$

$$z(t) = \cos[n(t-t_0)]\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega} + \sin[n(t-t_0)]\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega} \tag{36}$$

The parametric equations of the velocity become

$$\dot{x}(t) = -n\sin[n(t-t_0)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + n\cos[n(t-t_0)] \times \left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \cos[n(t-t_0)]\left\{-\omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} + \sin[n(t-t_0)] \times \left\{-\omega\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} \tag{37}$$

$$\dot{y}(t) = n\sin[n(t-t_0)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] - \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} + n\cos[n(t-t_0)] \times \left\{-\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} - \cos[n(t-t_0)]\left\{\frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right\} - \sin[n(t-t_0)] \times \left\{\frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right\} \tag{38}$$

$$\dot{z}(t) = -n\sin[n(t-t_0)]\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega} + n\cos[n(t-t_0)]\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega} \tag{39}$$

where $t \in [t_0, +\infty)$.

The projection of the trajectory on the three planes of the LVLH reference frame are Lissajous figures [18].

3. *The Periodicity Problem (Orbit Resonance)*

When $\boldsymbol{h}_0 \neq \boldsymbol{0}$, $\xi < 0$, the motion may be periodic if some conditions are satisfied. It is easy to see that the two functions involved in the expressions of motion are periodic: function $\boldsymbol{R}_{-\boldsymbol{\omega}}$ has the main period $2\pi/\omega$ and function E has the period $2\pi/n$. When the ratio ω/n is a rational number, the motion is periodic; this is a necessary and sufficient condition for periodicity. Some interesting situations may occur. We denote

$$\frac{\omega}{n} = \frac{k}{q} \tag{40}$$

with k and q natural relative prime numbers. The fraction k/q represents the ratio between the Deputy period and the Chief period.

Denoting ξ_C the energy of the Chief satellite, $\xi_C = -\omega^2 r_0^2$, relation (40) can be written as

$$\frac{\mu\sqrt{|\xi_C|}}{r_0(2|\xi|)^{3/2}} = \frac{k}{q} \Leftrightarrow \frac{\mu\omega}{(2|\xi|)^{3/2}} = \frac{k}{q} \tag{41}$$

Expressed in terms of orbital elements (r_0 the radius of the circular reference orbit and a the semimajor axis of the Deputy elliptic trajectory), Eq. (41) is equivalent to

$$\frac{a}{r_0} = \left(\frac{k}{q}\right)^{2/3} \tag{42}$$

where k and n are relative prime natural numbers.

The equivalent Eqs. (41) and (42) represent the necessary and sufficient condition for the trajectory to be closed (the relative motion is periodic). This particular situation is called *orbit resonance* [23,25,32].

Related to an inertial reference frame centered in the attraction point O, three important cases occur.

Case $k = 1$, $q > 1$: the Chief completes one circle and the Deputy q ellipses (or circles, depending on e) until the configuration Chief–Deputy has the same position as at $t = t_0$.

Case $k > 1$, $q = 1$: the Chief completes k circles and the deputy a single ellipse (circle) until the configuration Chief–Deputy has the same position as at $t = t_0$.

Case $k = q = 1$: the Chief and the Deputy complete both an entire trajectory until they meet as at $t = t_0$.

4. Zero Angular Momentum: $\boldsymbol{h}_0 = \boldsymbol{0}$

According to [64], Subsection IV.B.2, it follows that

$$\boldsymbol{a}_0 = \frac{\mu}{2|\xi|}\frac{\boldsymbol{r}_0 + \Delta\boldsymbol{r}}{|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|} \tag{43}$$

$$\boldsymbol{r}(t) = [\cos E(t) - 1]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} - \boldsymbol{r}_0 \tag{44}$$

$$\boldsymbol{v}(t) = \frac{-n\sin E(t)}{[1-\cos E(t)]}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + [\cos E(t) - 1] \times\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0\right\} \tag{45}$$

where

$$\begin{aligned} E(t) - \sin E(t) &= n(t-t_0) + E_0 - \sin E_0 \\ \cos E_0 &= 1 - \frac{2|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|\cdot|\xi|}{\mu} \\ \sin E_0 &= \frac{\sqrt{2|\xi|}\Delta\boldsymbol{v}\cdot(\boldsymbol{r}_0 + \Delta\boldsymbol{r})}{\mu} \end{aligned} \tag{46}$$

$$n = \frac{(2|\xi|)^{3/2}}{\mu}; \qquad t_P = t_0 - \frac{1}{n}(E_0 - \sin E_0); \qquad t\in\left[t_0, t_P + \frac{2\pi}{n}\right) \tag{47}$$

The parametric equations of the trajectory become

$$x(t) = [\cos E(t) - 1]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} - r_0 \tag{48}$$

$$y(t) = [\cos E(t) - 1]\left\{-\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} \tag{49}$$

$$z(t) = [\cos E(t) - 1]\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega} \tag{50}$$

The parametric equations of the velocity become

$$\dot{x}(t) = \frac{-n\sin E(t)}{1-\cos E(t)}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + [\cos E(t) - 1] \times\left\{-\omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} \tag{51}$$

$$\dot{y}(t) = \frac{n\sin E(t)}{1-\cos E(t)}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] - \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} - [\cos E(t) - 1] \times\left\{\frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right\} \tag{52}$$

$$\dot{z}(t) = \frac{-n\sin E(t)}{1-\cos E(t)}\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega} \tag{53}$$

In this case, theoretically, the motion lasts until the body reaches the attraction point. Practically, the body falls on the planet, so the motion lasts on the time interval $[t_0, t_1]$, where t_1 denotes the ground-impact moment, $t_1 < t_P + 2\pi/n$.

B. Zero Specific Energy: $\boldsymbol{\xi = \frac{1}{2}[\Delta v + \omega\times(\Delta r + r_0)]^2 - \frac{\mu}{|\Delta r + r_0|} = 0}$

1. Nonzero Angular Momentum: $\boldsymbol{h}_0 \neq \boldsymbol{0}$

Applying the results from [64], Subsection IV.B.2, it results the law of motion

$$\boldsymbol{r}(t) = \frac{1}{2}[p - \mu\tau^2(t)]\left\{\frac{\boldsymbol{e}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{e}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{e}_0\right\} + \tau(t)\left\{\frac{(\boldsymbol{h}_0,\boldsymbol{e}_0,\boldsymbol{\omega})}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}(\boldsymbol{h}_0\times\boldsymbol{e}_0) - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2(\boldsymbol{h}_0\times\boldsymbol{e}_0)\right\} - \boldsymbol{r}_0 \tag{54}$$

and the velocity

$$\begin{aligned}\boldsymbol{v}(t) = {} & \frac{-2\mu\tau(t)}{p+\mu\tau^2(t)}\left\{\frac{\boldsymbol{e}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{e}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{e}_0\right\} + \frac{2}{p+\mu\tau^2(t)}\left\{\frac{(\boldsymbol{h}_0,\boldsymbol{e}_0,\boldsymbol{\omega})}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}(\boldsymbol{h}_0\times\boldsymbol{e}_0) - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2(\boldsymbol{h}_0\times\boldsymbol{e}_0)\right\} \\ & + \frac{1}{2}[p-\mu\tau^2(t)]\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{e}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{e}_0\right\} + \tau(t)\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2(\boldsymbol{h}_0\times\boldsymbol{e}_0) - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}(\boldsymbol{h}_0\times\boldsymbol{e}_0)\right\}\end{aligned} \tag{55}$$

where

$$\begin{aligned} \tau(t) &= \frac{1}{\sqrt[3]{\mu}}\sqrt[3]{3(t_P - t) + \sqrt{9(t_P-t)^2 + p^3/\mu}} + \frac{1}{\sqrt[3]{\mu}}\sqrt[3]{3(t_P - t) - \sqrt{9(t_P-t)^2 + p^3/\mu}}, \\ t_P &= t_0 - \frac{1}{2}\left[p\tau(t_0) + \frac{\mu}{3}\tau^3(t_0)\right], \\ \tau(t_0) &= \frac{(\Delta\boldsymbol{r} + \boldsymbol{r}_0)\cdot\Delta\boldsymbol{v}}{\mu^2}[\mu - |\Delta\boldsymbol{r} + \boldsymbol{r}_0|(\boldsymbol{\omega}_0\cdot\boldsymbol{h}_0)]; \\ t &\in [t_0, +\infty) \end{aligned} \tag{56}$$

In the LVLH reference frame the parametric equations of the trajectory are

$$x(t) = \frac{1}{2}[p - \mu\tau^2(t)]\left\{\frac{\boldsymbol{e}_0 \cdot \boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \tau(t)\left\{\frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{r}_0)}{r_0} \times \cos[\omega(t-t_0)] + \frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} - r_0 \quad (57)$$

$$y(t) = \frac{1}{2}[p - \mu\tau^2(t)]\left\{-\frac{\boldsymbol{e}_0 \cdot \boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} + \tau(t)\left\{\frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega r_0} \times \cos[\omega(t-t_0)] - \frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)]\right\} \quad (58)$$

$$z(t) = \frac{1}{2}[p - \mu\tau^2(t)]\frac{\boldsymbol{e}_0 \cdot \boldsymbol{\omega}}{\omega} + \tau(t)\frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{\omega})}{\omega} \quad (59)$$

The parametric equations of the velocity are

$$\dot{x}(t) = \frac{-2\mu\tau(t)}{p + \mu\tau^2(t)}\left\{\frac{\boldsymbol{e}_0 \cdot \boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} \times \sin[\omega(t-t_0)]\right\} + \frac{2}{p + \mu\tau^2(t)}\left\{\frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \frac{1}{2}[p - \mu\tau^2(t)] \times \left\{-\omega\frac{\boldsymbol{e}_0 \cdot \boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} + \tau(t)\left\{-\omega\frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{h}_0 \times \boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} \quad (60)$$

$$\dot{y}(t) = \frac{2\mu\tau(t)}{p + \mu\tau^2(t)}\left\{\frac{\boldsymbol{e}_0 \cdot \boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] - \frac{(\boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} \times \cos[\omega(t-t_0)]\right\} + \frac{2}{p + \mu\tau^2(t)}\left\{\frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega r_0} \times \cos[\omega(t-t_0)] - \frac{(\boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \frac{1}{2}[p - \mu\tau^2(t)]\left\{\frac{(\boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{e}_0 \cdot \boldsymbol{r}_0}{r_0} \times \cos[\omega(t-t_0)]\right\} + \tau(t)\left\{\frac{(\boldsymbol{h}_0 \times \boldsymbol{e}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\} \quad (61)$$

$$\dot{z}(t) = \frac{2}{p + \mu\tau^2(t)}\left[-\mu\tau(t)\frac{\boldsymbol{e}_0 \cdot \boldsymbol{\omega}}{\omega} + \frac{(\boldsymbol{h}_0, \boldsymbol{e}_0, \boldsymbol{\omega})}{\omega}\right] \quad (62)$$

2. Zero Angular Momentum: $\boldsymbol{h}_0 = \boldsymbol{0}$

All computations in the preceding section change by making $\boldsymbol{h}_0 = \boldsymbol{0}$:

$$\boldsymbol{r}(t) = \frac{\mu^{1/3}[6(t-t_P)]^{2/3}}{2|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\left\{\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}(\boldsymbol{r}_0 + \Delta\boldsymbol{r}) - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\right\} - \boldsymbol{r}_0 \quad (63)$$

$$\boldsymbol{v}(t) = \frac{2}{|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\left[\frac{6(t-t_P)}{\mu}\right]^{-1/3}\left\{\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}(\boldsymbol{r}_0 + \Delta\boldsymbol{r}) - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\right\} + \frac{\mu^{1/3}[6(t-t_P)]^{2/3}}{2|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2(\boldsymbol{r}_0 + \Delta\boldsymbol{r}) - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\right\} \quad (64)$$

where

$$t_P = t_0 - \frac{[(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\Delta\boldsymbol{v}]^{3/2}}{6\mu^2} \quad (65)$$

Formulas (63) and (64) work for $t \in [t_0, t_P)$ if $(\Delta\boldsymbol{r} + \boldsymbol{r}_0)\cdot\Delta\boldsymbol{v} < 0$ and $t \in [t_0, +\infty)$ if $(\Delta\boldsymbol{r} + \boldsymbol{r}_0)\cdot\Delta\boldsymbol{v} > 0$.

In the LVLH frame, we obtain the parametric equations of the trajectory:

$$x(t) = \frac{\mu^{1/3}[6(t-t_P)]^{2/3}}{2}\left\{\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{r}_0}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\cos[\omega(t-t_0)] + \frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\sin[\omega(t-t_0)]\right\} - r_0 \quad (66)$$

$$y(t) = \frac{\mu^{1/3}[6(t-t_P)]^{2/3}}{2}\left\{-\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{r}_0}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\sin[\omega(t-t_0)] + \frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\cos[\omega(t-t_0)]\right\} \quad (67)$$

$$z(t) = \frac{\mu^{1/3}[6(t-t_P)]^{2/3}}{2}\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{\omega}}{\omega|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|} \quad (68)$$

The parametric equations of the velocity are

$$\dot{x}(t) = 2\left(\frac{\mu}{6(t-t_P)}\right)^{1/3}\left\{\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{r}_0}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\cos[\omega(t-t_0)] + \frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\sin[\omega(t-t_0)]\right\} + \frac{\mu^{1/3}[6(t-t_P)]^{2/3}}{2} \times \left\{-\omega\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{r}_0}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\sin[\omega(t-t_0)] + \frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\cos[\omega(t-t_0)]\right\} \quad (69)$$

$$\dot{y}(t) = 2\left(\frac{\mu}{6(t-t_P)}\right)^{1/3}\left\{-\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{r}_0}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\sin[\omega(t-t_0)] + \frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\cos[\omega(t-t_0)]\right\} - \frac{\mu^{1/3}[6(t-t_P)]^{2/3}}{2} \times \left\{\frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\sin[\omega(t-t_0)] + \omega\frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{r}_0}{r_0|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|}\cos[\omega(t-t_0)]\right\} \quad (70)$$

$$\dot{z}(t) = 2\left(\frac{\mu}{6(t-t_P)}\right)^{1/3} \frac{(\boldsymbol{r}_0 + \Delta\boldsymbol{r})\cdot\boldsymbol{\omega}}{\omega|\boldsymbol{r}_0 + \Delta\boldsymbol{r}|} \tag{71}$$

C. Positive Specific Energy: $\xi = \frac{1}{2}[\Delta\boldsymbol{v} + \boldsymbol{\omega}\times(\Delta\boldsymbol{r} + \boldsymbol{r}_0)]^2 - \frac{\mu}{|\Delta\boldsymbol{r}+\boldsymbol{r}_0|} > 0$

1. Nonzero Angular Momentum: $\boldsymbol{h}_0 \neq \boldsymbol{0}$

By using

$$\boldsymbol{a}_0 = \frac{\mu}{2e\xi}\boldsymbol{e}_0; \qquad \boldsymbol{b}_0 = \frac{1}{e\sqrt{2\xi}}\boldsymbol{h}_0\times\boldsymbol{e}_0; \qquad n = \frac{(2\xi)^{3/2}}{\mu} \tag{72}$$

and the function $E(t)$ given by

$$e\sinh E(t) - E(t) = n(t-t_0) + e\sinh E_0 - E_0, \qquad t\in[t_0,+\infty) \tag{73}$$

$$E_0 = \sinh^{-1}\left\{\frac{n[\Delta\boldsymbol{v}\cdot(\boldsymbol{r}_0+\Delta\boldsymbol{r})]}{2e\xi}\left[1 - \frac{\boldsymbol{\omega}_0\cdot\boldsymbol{h}_0}{\mu}|\boldsymbol{r}_0+\Delta\boldsymbol{r}|\right]\right\} \tag{74}$$

$$\begin{aligned}\boldsymbol{r}(t) &= [e-\cosh E(t)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + \sinh E(t)\\ &\times\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0\right\} - \boldsymbol{r}_0\end{aligned} \tag{75}$$

$$\begin{aligned}\boldsymbol{v}(t) &= \frac{-n\sinh E(t)}{e\cosh E(t)-1}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + \frac{n\cosh E(t)}{e\cosh E(t)-1}\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega}\right.\\ &\left.- \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0\right\} + [e - \cosh E(t)]\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0\right\}\\ &+ \sinh E(t)\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{b}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{b}_0\right\}\end{aligned} \tag{76}$$

where $t\in[t_0,+\infty)$.

In LVLH reference frame, the parametric equations of the trajectory are

$$\begin{aligned}x(t) &= [e-\cosh E(t)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \sinh E(t)\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right.\\ &\left.+ \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} - r_0\end{aligned} \tag{77}$$

$$\begin{aligned}y(t) &= [e-\cosh E(t)]\left\{-\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} + \sinh E(t)\\ &\times\left\{-\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\}\end{aligned} \tag{78}$$

$$z(t) = [e-\cosh E(t)]\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega} + \sinh E(t)\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega} \tag{79}$$

The parametric equations of the velocity are

$$\begin{aligned}\dot{x}(t) &= \frac{-n\sinh E(t)}{e\cosh E(t)-1}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + \frac{n\cosh E(t)}{e\cosh E(t)-1}\left\{\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right.\\ &\left.+ \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t-t_0)]\right\} + [e-\cosh E(t)]\times\left\{-\omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\}\\ &+ \sinh E(t)\left\{-\omega\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] + \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\cos[\omega(t-t_0)]\right\}\end{aligned} \tag{80}$$

$$\begin{aligned}\dot{y}(t) &= \frac{n\sinh E(t)}{e\cosh E(t)-1}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)] - \frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} + \frac{n\cosh E(t)}{e\cosh E(t)-1}\left\{-\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\sin[\omega(t-t_0)]\right.\\ &\left.+ \frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)]\right\} - [e-\cosh E(t)]\times\left\{\frac{(\boldsymbol{a}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{a}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right\}\\ &- \sinh E(t)\left\{\frac{(\boldsymbol{b}_0,\boldsymbol{\omega},\boldsymbol{r}_0)}{r_0}\sin[\omega(t-t_0)] + \omega\frac{\boldsymbol{b}_0\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)]\right\}\end{aligned} \tag{81}$$

$$\dot{z}(t) = \frac{-n\sinh E(t)}{e\cosh E(t)-1}\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega} + \frac{n\cosh E(t)}{e\cosh E(t)-1}\frac{\boldsymbol{b}_0\cdot\boldsymbol{\omega}}{\omega} \tag{82}$$

2. Zero Angular Momentum: $\boldsymbol{h}_0 = \boldsymbol{0}$

According to [64], Subsection IV.B.2,

$$\begin{aligned}&\boldsymbol{a}_0 = \frac{\mu}{2\xi}\frac{\boldsymbol{r}_0+\Delta\boldsymbol{r}}{|\boldsymbol{r}_0+\Delta\boldsymbol{r}|}; \qquad n = \frac{(2\xi)^{3/2}}{\mu},\\ &\sinh E(t) - E(t) = n(t-t_0) + \sinh E_0 - E_0,\\ &E_0 = \sinh^{-1}\left[\frac{n[\Delta\boldsymbol{v}\cdot(\boldsymbol{r}_0+\Delta\boldsymbol{r})]}{2\xi}\right], \qquad t_P = t_0 - \frac{1}{n}(\sinh E_0 - E_0)\end{aligned} \tag{83}$$

$$\boldsymbol{r}(t) = [1-\cosh E(t)]\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} - \boldsymbol{r}_0 \tag{84}$$

$$\begin{aligned}\boldsymbol{v}(t) &= \frac{-n\sinh E(t)}{\cosh E(t)-1}\left\{\frac{\boldsymbol{a}_0\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0 - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0\right\} + [1-\cosh E(t)]\\ &\times\left\{\frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}}^2\boldsymbol{a}_0 - \cos[\omega(t-t_0)]\tilde{\boldsymbol{\omega}}\boldsymbol{a}_0\right\}\end{aligned} \tag{85}$$

where $t\in[t_0,t_P)$ if $(\Delta\boldsymbol{r}+\boldsymbol{r}_0)\cdot\Delta\boldsymbol{v} < 0$ and $t\in[t_0,+\infty)$ if $(\Delta\boldsymbol{r}+\boldsymbol{r}_0)\cdot\Delta\boldsymbol{v} > 0$.

In the LVLH reference frame, the parametric equations of the trajectory are

$$x(t) = [1 - \cosh E(t)]\left\{\frac{\boldsymbol{a}_0 \cdot \boldsymbol{r}_0}{r_0}\cos[\omega(t - t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\sin[\omega(t - t_0)]\right\} - r_0 \tag{86}$$

$$y(t) = [1 - \cosh E(t)]\left\{-\frac{\boldsymbol{a}_0 \cdot \boldsymbol{r}_0}{r_0}\sin[\omega(t - t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t - t_0)]\right\} \tag{87}$$

$$z(t) = [1 - \cosh E(t)]\frac{\boldsymbol{a}_0 \cdot \boldsymbol{\omega}}{\omega} \tag{88}$$

The parametric equations of the velocity are

$$\dot{x}(t) = \frac{-n \sinh E(t)}{\cosh E(t) - 1}\left\{\frac{\boldsymbol{a}_0 \cdot \boldsymbol{r}_0}{r_0}\cos[\omega(t - t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} \times \sin[\omega(t - t_0)]\right\} + [1 - \cosh E(t)]\left\{-\omega\frac{\boldsymbol{a}_0 \cdot \boldsymbol{r}_0}{r_0}\sin[\omega(t - t_0)] + \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\cos[\omega(t - t_0)]\right\} \tag{89}$$

$$\dot{y}(t) = \frac{n \sinh E(t)}{\cosh E(t) - 1}\left\{\frac{\boldsymbol{a}_0 \cdot \boldsymbol{r}_0}{r_0}\sin[\omega(t - t_0)] - \frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} \times \cos[\omega(t - t_0)]\right\} - [1 - \cosh E(t)]\left\{\frac{(\boldsymbol{a}_0, \boldsymbol{\omega}, \boldsymbol{r}_0)}{r_0}\sin[\omega(t - t_0)] + \omega\frac{\boldsymbol{a}_0 \cdot \boldsymbol{r}_0}{r_0}\cos[\omega(t - t_0)]\right\} \tag{90}$$

$$\dot{z}(t) = \frac{-n \sinh E(t)}{\cosh E(t) - 1}\frac{\boldsymbol{a}_0 \cdot \boldsymbol{\omega}}{\omega} \tag{91}$$

Remark 2: The closed-form vectorial solutions to the orbital relative motion problem as well as the parametric equations were deduced with respect to the noninertial LVLH reference frame. The preceding study is made when the reference trajectory is circular. The solutions have the time as independent variable and no singularities. In Subsection III.A.3 the necessary and sufficient conditions for the periodicity of the relative motion were introduced.

By using various expansions of map $E = E(M)$, where $M = n(t - t_P)$ is similar to the mean anomaly [6], various approximate expressions having a high degree of accuracy may be deduced with the time as independent variable.

The next Section proves that the solution to the famous Hill–Clohessy–Wiltshire equations is the first linear approximation to the exact solution presented in Subsection III.A.2.

D. About Hill–Clohessy–Wiltshire Equations

A typical approach to relative orbital motion are the Hill–Clohessy–Wiltshire (HCW) equations. They offer an approximate solution for a circular reference orbit. By using the tensorial instrument introduced in [64], Sec. II, we give an explicit exact vectorial solution to the problem.

The differential equation

$$\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \frac{\mu}{\rho^3}\boldsymbol{\rho} = \boldsymbol{0}, \qquad \boldsymbol{\rho}(t_0) = \boldsymbol{r}_0 + \Delta\boldsymbol{r}, \qquad \dot{\boldsymbol{\rho}}(t_0) = \Delta\boldsymbol{v} \tag{92}$$

with the additional conditions

$$r_0 = \left(\frac{\mu}{\omega^2}\right)^{1/3}; \qquad \boldsymbol{r}_0 \cdot \boldsymbol{\omega} = 0; \qquad \boldsymbol{\omega} = \overrightarrow{\text{constant}}$$

has a vectorial map $\boldsymbol{\rho} = \boldsymbol{\rho}(t, \boldsymbol{r}_0, \Delta\boldsymbol{r}, \Delta\boldsymbol{v})$ as solution. For $\Delta\boldsymbol{r} = \boldsymbol{0}$ and $\Delta\boldsymbol{v} = \boldsymbol{0}$, it has the constant solution $\boldsymbol{\rho}(t) = \boldsymbol{r}_0$, $t \in [0, +\infty)$, and it describes the Chief motion in its associated reference frame with origin in the attraction center.

We prove now that the solution in first approximation to the perturbed Eq. (92) leads to the solution to HCW equations, used in numerous papers [1–9,65].

Seeking $\boldsymbol{\rho} = \boldsymbol{r}_0 + \boldsymbol{r}$, vector $\boldsymbol{r}$ represents the relative position of the Deputy. Replacing in (92) it follows that

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_0) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \frac{\mu}{|\boldsymbol{r}_0 + \boldsymbol{r}|^3}(\boldsymbol{r}_0 + \boldsymbol{r}) = \boldsymbol{0}, \qquad \boldsymbol{r}(t_0) = \Delta\boldsymbol{r}, \qquad \dot{\boldsymbol{r}}(t_0) = \Delta\boldsymbol{v} \tag{93}$$

For $(r/r_0)^n \simeq 0$, $n \geq 2$ we make the following approximations:

$$\frac{\mu}{|\boldsymbol{r}_0 + \boldsymbol{r}|^3} = \mu(\boldsymbol{r}_0^2 + 2\boldsymbol{r}_0 \cdot \boldsymbol{r} + \boldsymbol{r}^2)^{-3/2} \simeq \mu\boldsymbol{r}_0^{-3}\left(1 + \frac{2\boldsymbol{r}_0 \cdot \boldsymbol{r}}{\boldsymbol{r}_0^2}\right)^{-3/2} \simeq \mu\boldsymbol{r}_0^{-3}\left(1 - \frac{3\boldsymbol{r}_0 \cdot \boldsymbol{r}}{\boldsymbol{r}_0^2}\right) \tag{94}$$

and take into account that $\mu r_0^{-3} = \omega^2$. This leads to a new problem for $\boldsymbol{r}$:

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \omega^2\boldsymbol{r} = 3\omega^2\frac{\boldsymbol{r} \cdot \boldsymbol{r}_0}{r_0^2}\boldsymbol{r}_0, \qquad \boldsymbol{r}(t_0) = \Delta\boldsymbol{r}, \qquad \dot{\boldsymbol{r}}(t_0) = \Delta\boldsymbol{v} \tag{95}$$

Equation (95) is the vectorial form of the HCW equations. It models the relative orbiting motion. It is a differential vectorial linear equation. The tensorial proper orthogonal map $\boldsymbol{R}_{\boldsymbol{\omega}}$ is defined in [64], Section II.A.

Lemma 1: The solution to Eq. (95) is obtained by applying operator $\boldsymbol{R}_{-\boldsymbol{\omega}}$ to the solution to the differential equation

$$\ddot{\boldsymbol{r}} + \omega^2\boldsymbol{r} = 3\omega^2(\boldsymbol{i} \otimes \boldsymbol{i})\boldsymbol{r}, \qquad \boldsymbol{r}(t_0) = \Delta\boldsymbol{r}, \qquad \dot{\boldsymbol{r}}(t_0) = \Delta\boldsymbol{v} + \boldsymbol{\omega} \times \Delta\boldsymbol{r} \tag{96}$$

where $\otimes$ represents the dyadic product of two vectors and $\boldsymbol{i} = \boldsymbol{R}_{\boldsymbol{\omega}} i_0 = \boldsymbol{R}_{\boldsymbol{\omega}}(\boldsymbol{r}_0/r_0)$.

Proof: We notice that Eq. (95) can be rewritten in the form

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \omega^2\boldsymbol{r} = 3\omega^2(\boldsymbol{i}_0 \otimes \boldsymbol{i}_0)\boldsymbol{r} \tag{97}$$

As $\boldsymbol{R}_{\boldsymbol{\omega}}[(\boldsymbol{i}_0 \otimes \boldsymbol{i}_0)\boldsymbol{r}] = (\boldsymbol{i} \otimes \boldsymbol{i})\boldsymbol{R}_{\boldsymbol{\omega}}\boldsymbol{r}$, applying operator $\boldsymbol{R}_{\boldsymbol{\omega}}$ to relation (97) and using its properties, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\boldsymbol{R}_{\boldsymbol{\omega}}\boldsymbol{r}) + \omega^2\boldsymbol{R}_{\boldsymbol{\omega}}\boldsymbol{r} = 3\omega^2(\boldsymbol{i} \otimes \boldsymbol{i})\boldsymbol{R}_{\boldsymbol{\omega}}\boldsymbol{r}$$

The proof is finalized. □

The way of obtaining the solution to Eqs. (95) and (96) and their qualitative analysis are given in the Appendix.

From relations (A16) and (A17), vectors $\boldsymbol{r}$ and $\boldsymbol{v}$ may be written as

$$\boldsymbol{r} = \boldsymbol{T}_{11}\Delta\boldsymbol{r} + \boldsymbol{T}_{12}\Delta\boldsymbol{v}, \qquad \boldsymbol{v} = \boldsymbol{T}_{21}\Delta\boldsymbol{r} + \boldsymbol{T}_{22}\Delta\boldsymbol{v} \tag{98}$$

where $\boldsymbol{T}_{ij} = \boldsymbol{T}_{ij}(t, \boldsymbol{r}_0, \boldsymbol{\omega})$, $i, j \in \{1, 2\}$ are tensorial maps given by

$$\boldsymbol{T}_{11} = 4\frac{\boldsymbol{r}_0 \otimes \boldsymbol{r}_0}{r_0^2} + \frac{(\boldsymbol{\omega} \times \boldsymbol{r}_0) \otimes (\boldsymbol{\omega} \times \boldsymbol{r}_0)}{\omega^2 r_0^2} - 6t\frac{\boldsymbol{r}_0 \otimes (\boldsymbol{\omega} \times \boldsymbol{r}_0)}{r_0^2} + 6\frac{\boldsymbol{r}_0 \otimes (\boldsymbol{\omega} \times \boldsymbol{r}_0)}{\omega r_0^2}\sin[\omega(t - t_0)] + \left[-3\frac{\boldsymbol{r}_0 \otimes \boldsymbol{r}_0}{r_0^2} + \frac{\boldsymbol{\omega} \otimes \boldsymbol{\omega}}{\omega^2}\right] \times \cos[\omega(t - t_0)] \tag{99}$$

$$\begin{aligned}T_{12} &= 2\frac{(\boldsymbol{\omega}\times\boldsymbol{r}_0)\otimes\boldsymbol{r}_0}{\omega^2 r_0^2} - 2\frac{\boldsymbol{r}_0\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega^2 r_0^2} - 3t\frac{\boldsymbol{r}_0\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{r_0^2}\\ &+ \left[\frac{(\boldsymbol{\omega}\times\boldsymbol{r}_0)\otimes\boldsymbol{r}_0}{\omega^2 r_0^2} + 4\frac{(\boldsymbol{\omega}\times\boldsymbol{r}_0)\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega^3 r_0^2} + \frac{\boldsymbol{\omega}\otimes\boldsymbol{\omega}}{\omega^3}\right]\\ &\times\sin[\omega(t-t_0)] + \left[2\frac{\boldsymbol{r}_0\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega^2 r_0^2} - 2\frac{(\boldsymbol{\omega}\times\boldsymbol{r}_0)\otimes\boldsymbol{r}_0}{\omega^2 r_0^2}\right]\\ &\times\cos[\omega(t-t_0)]\end{aligned} \tag{100}$$

$$\begin{aligned}T_{21} &= -6\frac{\boldsymbol{r}_0\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{r_0^2} + 6\frac{\boldsymbol{r}_0\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{r_0^2}\cos[\omega(t-t_0)]\\ &+ \left[3\frac{\boldsymbol{r}_0\otimes\boldsymbol{r}_0}{r_0^2} - \frac{\boldsymbol{\omega}\otimes\boldsymbol{\omega}}{\omega^2}\right]\sin[\omega(t-t_0)]\end{aligned} \tag{101}$$

$$\begin{aligned}T_{22} &= -3\frac{\boldsymbol{r}_0\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{r_0^2} + \left[\frac{(\boldsymbol{\omega}\times\boldsymbol{r}_0)\otimes\boldsymbol{r}_0}{\omega r_0^2}\right.\\ &\left.+ 4\frac{(\boldsymbol{\omega}\times\boldsymbol{r}_0)\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega^2 r_0^2} + \frac{\boldsymbol{\omega}\otimes\boldsymbol{\omega}}{\omega^2}\right]\cos[\omega(t-t_0)]\\ &- \left[2\frac{\boldsymbol{r}_0\otimes(\boldsymbol{\omega}\times\boldsymbol{r}_0)}{\omega r_0^2} - 2\frac{(\boldsymbol{\omega}\times\boldsymbol{r}_0)\otimes\boldsymbol{r}_0}{\omega r_0^2}\right]\sin[\omega(t-t_0)]\end{aligned} \tag{102}$$

By denoting

$$T = \begin{bmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix} \tag{103}$$

the solution to Eq. (104) may be written in the form

$$\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{v}\end{bmatrix} = T\begin{bmatrix}\Delta\boldsymbol{r}\\ \Delta\boldsymbol{v}\end{bmatrix} \tag{104}$$

In Eq. (104), T represents the state transition matrix [3] of the relative Keplerian motion.

The first linear approximation of the solution to Eq. (92) is written as

$$\boldsymbol{\rho} = \boldsymbol{r}_0 + T_{11}\Delta\boldsymbol{r} + T_{12}\Delta\boldsymbol{v}, \qquad \dot{\boldsymbol{\rho}} = T_{21}\Delta\boldsymbol{r} + T_{12}\Delta\boldsymbol{v} \tag{105}$$

By denoting

$$X = \begin{bmatrix}\boldsymbol{\rho}\\ \dot{\boldsymbol{\rho}}\end{bmatrix}, \qquad X_0 = \begin{bmatrix}\boldsymbol{r}_0\\ \boldsymbol{0}\end{bmatrix}, \qquad \Delta X_0 = \begin{bmatrix}\Delta\boldsymbol{r}\\ \Delta\boldsymbol{v}\end{bmatrix}$$

the following relation holds true:

$$X = X_0 + T\Delta X_0 \tag{106}$$

Equation (106) is the first approximation to the solution to the Cauchy problem that describes the relative motion of the Deputy. It holds true only when the approximation $(r/r_0)^n \simeq 0$, $n \geq 2$ can be used.

IV. Conclusions

A general exact solution to the relative orbital motion problem was offered for elliptic, parabolic, and hyperbolic reference trajectory. The result is purely vectorial and it is obtained by using an adequate tensorial instrument. The circular reference orbit problem is solved and explicit expressions for position and velocity vectors are given. The only implicit function involved in the exact solutions offered is analogous to the eccentric anomaly from the classic Keplerian motion. The results appear in a vectorial closed form. A vectorial exact solution to Hill–Clohessy–Wiltshire equations was given by using the tensorial instrument introduced, together with an interesting interpretation of the results that were obtained. The model offered in solving the relative motion problem may be applied to an arbitrary Keplerian reference orbit.

Appendix: A Vectorial Solution to the Hill–Clohessy–Wiltshire Equations

By using the tensorial method introduced in [64], the solution to the differential equation

$$\begin{aligned}\ddot{\boldsymbol{r}} + \omega^2\boldsymbol{r} = 3\omega^2(\boldsymbol{i}\otimes\boldsymbol{i})\boldsymbol{r}, \qquad \boldsymbol{r}(t_0) = \Delta\boldsymbol{r}\\ \dot{\boldsymbol{r}}(t_0) = \Delta\boldsymbol{v} + \boldsymbol{\omega}\times\Delta\boldsymbol{r}\end{aligned} \tag{A1}$$

is deduced.

Because of the linearity of Eq. (A1), its solution is the sum of the solutions to the following equations:

$$\begin{aligned}\dot{\boldsymbol{r}} + \omega^2\boldsymbol{r} = 3\omega^2(\boldsymbol{i}\otimes\boldsymbol{i})\boldsymbol{r}, \qquad \boldsymbol{r}(t_0) = \frac{\Delta\boldsymbol{r}\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega}\\ \dot{\boldsymbol{r}}(t_0) = \frac{\Delta\boldsymbol{v}\cdot\boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega}\end{aligned} \tag{A2}$$

$$\begin{aligned}\ddot{\boldsymbol{r}} + \omega^2\boldsymbol{r} = 3\omega^2(\boldsymbol{i}\otimes\boldsymbol{i}), \qquad \boldsymbol{r}(t_0) = \frac{\boldsymbol{\omega}\times(\Delta\boldsymbol{r}\times\boldsymbol{\omega})}{\omega^2}\\ \dot{\boldsymbol{r}}(t_0) = \frac{\boldsymbol{\omega}\times[(\Delta\boldsymbol{v}+\boldsymbol{\omega}\times\Delta\boldsymbol{r})\times\boldsymbol{\omega}]}{\omega^2}\end{aligned} \tag{A3}$$

The solution to Eq. (A2) is

$$\boldsymbol{r}_1(t) = \left\{\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{\omega}}{\omega}\cos[\omega(t-t_0)] + \frac{\Delta\boldsymbol{v}\cdot\boldsymbol{\omega}}{\omega^2}\sin[\omega(t-t_0)]\right\}\boldsymbol{\omega} \tag{A4}$$

We denote

$$\boldsymbol{j} = \frac{\boldsymbol{\omega}\times\boldsymbol{R}_\omega\boldsymbol{r}_0}{\omega r_0} \tag{A5}$$

The solution to Eq. (A3) is

$$\boldsymbol{r}_2(t) = \alpha(t)\boldsymbol{i} + \beta(t)\boldsymbol{j} \tag{A6}$$

where maps α and β are solutions to the ordinary differential equations (ODE)

$$\ddot{\alpha} - 2\omega\beta = 3\omega^2\alpha; \qquad \ddot{\beta} + 2\omega\dot{\alpha} = 0 \tag{A7}$$

with the initial values

$$\begin{aligned}\alpha(t_0) = \frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0^2}\boldsymbol{r}_0, \qquad \dot{\alpha}(0) = \frac{\Delta\boldsymbol{v}\cdot\boldsymbol{r}_0}{r_0^2}\boldsymbol{r}_0\\ \beta(t_0) = \frac{(\Delta\boldsymbol{r},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0^2}\boldsymbol{r}_0, \qquad \dot{\beta}(0) = \frac{(\Delta\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0^2}\boldsymbol{r}_0\end{aligned} \tag{A8}$$

The solution to the ODE system (A7) is

$$\begin{aligned}\alpha(t) &= 2\frac{(\Delta\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega^2 r_0} + 4\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + \frac{(\Delta\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega^2 r_0}\sin[\omega(t-t_0)]\\ &- \left(3\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 2\frac{(\Delta\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega^2 r_0}\right)\cos[\omega(t-t_0)],\\ \beta(t) &= \frac{(\Delta\boldsymbol{r},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0} - 2\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{\omega r_0} - \left(3\frac{(\Delta\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega r_0}\right.\\ &\left.+ 6\omega\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0}\right)(t-t_0) + \left(6\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 4\frac{(\Delta\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{r}_0)}{\omega^2 r_0}\right)\\ &\times\sin[\omega(t-t_0)] + 2\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{\omega r_0}\cos[\omega(t-t_0)]\end{aligned} \tag{A9}$$

The solution to Eq. (A1) is

$$\boldsymbol{\rho}(t) = \alpha(t)\boldsymbol{i} + \beta(t)\boldsymbol{j} + \gamma(t)\frac{\boldsymbol{\omega}}{\omega} \tag{A10}$$

Because

$$\boldsymbol{i} = \boldsymbol{R}_\omega \frac{\boldsymbol{r}_0}{r_0}; \qquad \boldsymbol{j} = \frac{\boldsymbol{\omega} \times \boldsymbol{R}_\omega \boldsymbol{r}_0}{\omega r_0}; \qquad \boldsymbol{\omega} = \boldsymbol{R}_\omega \boldsymbol{\omega} \tag{A11}$$

it follows that the solution to Eq. (A1) may be written as

$$\boldsymbol{\rho}(t) = \boldsymbol{R}_\omega \left[\alpha(t)\frac{\boldsymbol{r}_0}{r_0} + \beta(t)\frac{\boldsymbol{\omega} \times \boldsymbol{r}_0}{\omega r_0} + \gamma(t)\frac{\boldsymbol{\omega}}{\omega}\right] \tag{A12}$$

Applying Lemma 1, it follows that

$$\boldsymbol{r}(t) = \alpha(t)\frac{\boldsymbol{r}_0}{r_0} + \beta(t)\frac{\boldsymbol{\omega} \times \boldsymbol{r}_0}{\omega r_0} + \gamma(t)\frac{\boldsymbol{\omega}}{\omega} \tag{A13}$$

is the solution to Eq. (95), the vectorial form of the HCW equations.

The parametric equations of the trajectory in LVLH frame are

$$\begin{aligned}
\alpha(t) &= 2\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0} + 4\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + \frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0}\sin[\omega(t-t_0)] \\
&\quad - \left(3\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 2\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0}\right)\cos[\omega(t-t_0)] \\
\beta(t) &= \frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} - 2\frac{\Delta\boldsymbol{v}\cdot\boldsymbol{r}_0}{\omega r_0} - \left(3\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\right. \\
&\quad \left. + 6\omega\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0}\right)(t-t_0) + \left(6\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 4\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0}\right) \\
&\quad \times \sin[\omega(t-t_0)] + 2\frac{\Delta\boldsymbol{v}\cdot\boldsymbol{r}_0}{\omega r_0}\cos[\omega(t-t_0)] \\
\gamma(t) &= \frac{\Delta\boldsymbol{r}\cdot\boldsymbol{\omega}}{\omega}\cos[\omega(t-t_0)] + \frac{\Delta\boldsymbol{v}\cdot\boldsymbol{\omega}}{\omega^2}\sin[\omega(t-t_0)]
\end{aligned} \tag{A14}$$

The parametric equations of the velocity in LVLH are

$$\begin{aligned}
\dot{\alpha}(t) &= \frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\cos[\omega(t-t_0)] \\
&\quad - \left(3\omega\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 2\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\right)\sin[\omega(t-t_0)], \\
\dot{\beta}(t) &= -\left(3\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} + 6\omega\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0}\right) \\
&\quad + \left(6\omega\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 4\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\right)\sin[\omega(t-t_0)] \\
&\quad - 2\frac{\Delta\boldsymbol{v}\cdot\boldsymbol{r}_0}{r_0}\cos[\omega(t-t_0)], \\
\dot{\gamma}(t) &= -\Delta\boldsymbol{r}\cdot\boldsymbol{\omega}\sin[\omega(t-t_0)] + \frac{\Delta\boldsymbol{v}\cdot\boldsymbol{\omega}}{\omega}\cos[\omega(t-t_0)]
\end{aligned} \tag{A15}$$

The exact solution to Eq. (95) is given by Eqs. (A13). Equations (A14) and (A15) describe the motion of the Deputy with respect to the Chief, expressed in the LVLH frame associated with the Chief.

The solution to Eq. (A13) may be written as

$$\boldsymbol{r}(t) = \boldsymbol{r}_C + (t-t_0)\boldsymbol{v}_C + \boldsymbol{d}\sin[\omega(t-t_0)] + \boldsymbol{d}^*\cos[\omega(t-t_0)] \tag{A16}$$

and it represents the law of motion of the Deputy with respect to the Chief.

The relative velocity of the Deputy has the vectorial expression

$$\boldsymbol{v}(t) = \boldsymbol{v}_C + \omega\boldsymbol{d}\cos[\omega(t-t_0)] - \omega\boldsymbol{d}^*\sin[\omega(t-t_0)] \tag{A17}$$

From relations (A13) and (A14), vectors $\boldsymbol{r}_C$, $\boldsymbol{v}_C$, $\boldsymbol{d}$ and $\boldsymbol{d}^*$ are given by

$$\begin{aligned}
\boldsymbol{r}_C &= \left[2\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0} + 4\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0}\right]\frac{\mathbf{r}_0}{r_0} + \left[\frac{(\Delta\boldsymbol{r}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0}\right. \\
&\quad \left. - 2\frac{\Delta\boldsymbol{v}\cdot\boldsymbol{r}_0}{\omega r_0}\right]\frac{\boldsymbol{\omega}\times\boldsymbol{r}_0}{\omega r_0}, \\
\boldsymbol{v}_C &= -\left(3\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} + 6\omega\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0}\right)\frac{\boldsymbol{\omega}\times\mathbf{r}_0}{\omega r_0}, \\
\boldsymbol{d} &= \frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0}\frac{\boldsymbol{r}_0}{r_0} + \left(6\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 4\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0}\right)\frac{\boldsymbol{\omega}\times\boldsymbol{r}_0}{\omega r_0} \\
&\quad + \frac{\Delta\boldsymbol{v}\cdot\boldsymbol{\omega}}{\omega^2}\frac{\boldsymbol{\omega}}{\omega}, \\
\boldsymbol{d}^* &= -\left(3\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} + 2\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega^2 r_0}\right)\frac{\boldsymbol{r}_0}{r_0} + \frac{\Delta\boldsymbol{v}\cdot\boldsymbol{r}_0}{\omega r_0}\frac{\boldsymbol{\omega}\times\boldsymbol{r}_0}{\omega r_0} \\
&\quad + \frac{\Delta\boldsymbol{r}\cdot\boldsymbol{\omega}}{\omega}\frac{\boldsymbol{\omega}}{\omega}
\end{aligned} \tag{A18}$$

The last terms of relation (A16), namely $\boldsymbol{d}\sin[\omega(t-t_0)] + \boldsymbol{d}^*\cos[\omega(t-t_0)]$ represent the parametric equation of an ellipse having $\boldsymbol{d}$ and $\boldsymbol{d}^*$ as conjugate diameters [66].

As an interesting remark, we notice that

$$\boldsymbol{v}_C = \frac{1}{T}\int_{t_0}^{t_0+T}\boldsymbol{v}(t)\,\mathrm{d}t \tag{A19}$$

represents the mean velocity on a time interval of length $T = 2\pi/\omega$. The motion is periodic if $\boldsymbol{v}_C = \mathbf{0}$. Relation

$$\frac{(\Delta\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{r}_0)}{\omega r_0} + 2\omega\frac{\Delta\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0} = 0 \tag{A20}$$

is obtained in this case.

Generally, the relative trajectory is an elliptical helix, possibly degenerated (if, for example, $\boldsymbol{d}\times\boldsymbol{d}^* = \mathbf{0}$). If $\boldsymbol{v}_C = \mathbf{0}$, the relative trajectory is an ellipse, possibly degenerated, or a circle, if $\boldsymbol{d}\cdot\boldsymbol{d}^* = \mathbf{0}$ and $|\boldsymbol{d}| = |\boldsymbol{d}^*|$. Its semiaxis A and B may be computed using Apollonius theorems:

$$A^2 + B^2 = d^2 + d^{*2}; \qquad A^2B^2 = (\boldsymbol{d}\times\boldsymbol{d})^2 \tag{A21}$$

and their expressions are

$$\begin{aligned}
A^2 &= \frac{1}{2}[(d^2 + d^{*2}) + \sqrt{(d^2 + d^{*2})^2 - 4(\boldsymbol{d}\times\boldsymbol{d}^*)^2}], \\
B^2 &= \frac{1}{2}[(d^2 + d^{*2}) - \sqrt{(d^2 + d^{*2})^2 - 4(\boldsymbol{d}\times\boldsymbol{d}^*)^2}]
\end{aligned} \tag{A22}$$

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